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# Coconvex polynomial approximation

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**Abstract**

Let $f \in \mathbb{C}[-1,1]$ change its convexity finitely many times, in the interval. We are interested in estimating the degree of approximation of $f$ by polynomials, and by piecewise polynomials, which are coconvex with it, namely, polynomials and piecewise polynomials that change their convexity exactly at the points where $f$ does. We obtain Jackson-type estimates and summarize the positive and negative results in a truth-table as we have previously done for comonotone approximation.


*Keywords:* Coconvex polynomial approximation; Jackson estimates

## 1. Introduction

Let $f \in \mathbb{C}[-1,1]$ change its convexity finitely many times, say $s \geqslant 0$ times, in the interval. We are interested in estimating the degree of approximation of $f$ by polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where $f$ does.

In a recent survey [14] we have collected all known positive and negative results on monotone and comonotone approximation on a finite interval, by algebraic polynomials in the uniform norm (see also [11]). We have established complete truth tables for the validity of Jackson-type estimates, involving the ordinary $k$th

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moduli of smoothness of the $r$th derivative of a given monotone or piecewise monotone function, as well as estimates involving the Ditzian–Totik moduli of smoothness.

We intend here to obtain the analogous results for convex and coconvex approximation.

There are two main ingredients in the proofs of positive results. First one has to establish the existence of piecewise polynomials which are both coconvex with $f$ and sufficiently close to it, and second, to show that such piecewise polynomials may be well approximated by polynomials which are coconvex with them. The latter was the main contents of our recent paper [15]. Thus, we concentrate here on establishing the former and on drawing the final conclusions from having obtained the two needed ingredients.

In a forthcoming paper, we will show that if we relax the requirement on the piecewise polynomial, allowing it not to be coconvex with $f$ in small neighborhoods of the points of change of convexity of $f$, then we may secure a little better estimates. We call this type of approximation nearly coconvex approximation (cf. [12]).

Let $I := [-1, 1]$ and denote by $\mathbb{C} = \mathbb{C}^0$ and $\mathbb{C}^r$, respectively, the space of continuous functions, and that of $r$-times continuously differentiable functions on $I$, equipped with the uniform norm

$$\|f\| := \max_{x \in I} |f(x)|.$$

Denote by $\mathbb{Y}_s$, $s \in \mathbb{N}$, the set of all collections $Y_s := \{y_i\}_{i=1}^s$, such that $-1 < y_s < \cdots < y_1 < 1$, and for $s = 0$, we write $\mathbb{Y}_0 := \{\emptyset\}$. For later reference set $y_0 := 1$ and $y_{s+1} := -1$. Finally, let $\Delta^2(Y_s)$ denote the collection of all functions $f \in \mathbb{C}$ that change convexity at the set $Y_s$, and are convex in $[y_{1,1}]$, that is, $f$ is convex in $[y_{2i+1}, y_{2i}]$, $0 \leqslant i \leqslant [s/2]$, and it is concave in $[y_{2i}, y_{2i-1}]$, $1 \leqslant i \leqslant [(s+1)/2]$. In particular $\Delta^2 := \Delta^2(Y_0)$ is the set of convex functions on $I$.

We wish to approximate a general function $f \in \Delta^2(Y_s)$, by means of polynomials which are coconvex with $f$, that is, which belong to $\Delta^2(Y_s)$. We denote the degree of coconvex approximation by

$$E_n^{(2)}(f, Y_s) := \inf_{p_n \in \Pi_n \cap \Delta^2(Y_s)} \|f - p_n\|,$$

where $\Pi_n$ is the set of algebraic polynomials of degree not exceeding $n$. In particular, we denote $E_n^{(2)}(f) := E_n^{(2)}(f, Y_0)$, the degree of convex approximation.

We will construct continuous piecewise polynomials on the Chebyshev partition, that are coconvex with $f \in \Delta^2(Y_s)$, and approximate it well. Namely, given $n \in \mathbb{N}$, $n > 1$, we set $x_j := x_{j,n} := \cos(j\pi/n)$, $j = 0, \dots, n$, the Chebyshev partition of $[-1, 1]$, and we denote $I_j := I_{j,n} := [x_j, x_{j-1}]$, $j = 1, \dots, n$. Let $\Sigma_{k,n}$ be the collection of all continuous piecewise polynomials of degree $k - 1$, on the Chebyshev partition, that is, if $S \in \Sigma_{k,n}$, then

$$S|_{I_j} = p_j, \quad j = 1, \dots, n,$$

where $p_j \in \Pi_{k-1}$, and

$$p_j(x_j) = p_{j+1}(x_j), \quad j = 1, \dots, n-1.$$

Given $Y_s \in \mathbb{Y}_s$, let

$$O_i := O_{i,n}(Y_s) := (x_{j+1}, x_{j-2}), \quad \text{if} \quad y_i \in [x_j, x_{j-1}),$$

where $x_{n+1} := -1$, $x_{-1} := 1$, and denote

$$O = O(n, Y_s) := \bigcup_{i=1}^{s} O_i, \quad O(n, \emptyset) := \emptyset.$$

Finally, we write $j \in H = H(n, Y_s)$, if $I_j \cap O = \emptyset$, and denote by $\Sigma_{k,n}(Y_s) \subseteq \Sigma_{k,n}$, the subset of those piecewise polynomials for which

$$p_j \equiv p_{j+1}, \quad \text{whenever both } j, (j+1) \notin H.$$

The following result has been proved recently by Leviatan and Shevchuk [15].

**Theorem LS.** *For every $k \in \mathbb{N}$ and $s \in \mathbb{N}_0$ there are constants $c = c(k, s)$ and $c_* = c_*(k, s)$, such that if $n \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$, and $S \in \Sigma_{k,n}(Y_s) \cap \Delta^2(Y_s)$, then there is a polynomial $P_n \in \Delta^2(Y_s)$ of degree $\leqslant c_* n$, satisfying*

$$\|S - P_n\| \leqslant c\omega_k^{\varphi}(S, 1/n). \tag{1.1}$$

(For the definition of $\omega_k^{\varphi}(f, t)$, see Section 2.) Thus, if we are able to construct a good piecewise polynomial approximation, of the above type, to $f \in \Delta^2(Y_s)$, then we will have a good polynomial approximation to $f$.

In Section 2 we prove some auxiliary lemmas. In Section 3 we discuss convex approximation, and Section 4 is devoted to coconvex approximation.

In the sequel we will have absolute positive constants $C$, and we will have positive constants $c$ that depend only on $s$, $k$ and $r$, that are going to be indicated. We will use the notation $C$ and $c$ for such constants which are of no significance to us and may differ on different occurrences, even in the same line.

## 2. Auxiliary lemmas

In this section we collect some known results as well as new lemmas. In addition to the spaces of continuously differentiable functions we need two additional spaces. We will use the norm

$$\|f\| := \operatorname*{esssup}_{x \in I} |f(x)|,$$

also for a function that is essentially bounded on $I$, and with this notation, let the space $W^r$, be the set of functions $f \in \mathbb{C}$ which possess an absolutely continuous $(r-1)$st derivative in $I$, such that $\|f^{(r)}\| < \infty$. Also let the space $B^r$, be the set of functions $f \in \mathbb{C}$ which possess a locally absolutely continuous $(r-1)$st derivative in $(-1, 1)$, such that $\|\varphi^r f^{(r)}\| < \infty$, where $\varphi(x) := \sqrt{1 - x^2}$.

We sometimes wish to restrict ourselves to a subinterval $[a,b]\subseteq I$ in which case we will use the notation $\|\cdot\|_{[a,b]}$ for the above norms on the interval $[a,b]$. Then given $f\in\mathbb{C}[a,b]$, and $k\in\mathbb{N}$, we let

$$\Delta_h^k f(x) := \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f\left(x-\frac{k}{2}h+ih\right),$$

be the symmetric difference of order $k$, defined for all $x$ and $h\geqslant 0$, such that $x\pm\frac{k}{2}h\in[a,b]$. The ordinary moduli of smoothness of $f$ in $[a,b]$, $\omega_k(f,t;[a,b])$, are defined by

$$\omega_k(f,t;[a,b]) := \sup_{0\leqslant h\leqslant t}\sup_x |\Delta_h^k f(x)|, \quad t\geqslant 0,$$

where the inner supremum is taken over all $x$ such that $x\pm\frac{k}{2}h\in[a,b]$. In particular when $[a,b]=I$, we write $\omega_k(f,t):=\omega_k(f,t;I)$. We also need the Ditzian–Totik (DT-)moduli of smoothness [2] which on $[a,b]$ are defined by

$$\omega_k^{\phi}(f,t;[a,b]) := \sup_{0\leqslant h\leqslant t}\sup_x |\Delta_{h\phi(x)}^k f(x)|, \quad t\geqslant 0,$$

where $\phi(x):=\sqrt{(b-x)(x-a)}$ and the inner supremum is taken over all $x$ such that $x\pm\frac{k}{2}h\phi(x)\in[a,b]$. In particular for $I$, we have $\phi=\varphi$ and we denote $\omega_k^{\varphi}(f,t):=\omega_k^{\varphi}(f,t;I)$. It is well known that

$$\omega_k^{\varphi}(f,t)\leqslant c(k)\omega_k(f,t), \quad t>0.$$

If $f\in\mathbb{C}^r$, then

$$\omega_k(f,t)\leqslant c(k,r)t^r\omega_{k-r}(f^{(r)},t), \quad t>0, \quad k>r \tag{2.1}$$

and

$$\omega_k^{\varphi}(f,t)\leqslant c(k,r)t^r\omega_{k-r}^{\varphi}(f^{(r)},t), \quad t>0, \quad k>r. \tag{2.2}$$

Also if $f\in W^r$, then

$$\omega_r(f,t)\leqslant c(r)t^r\|f^{(r)}\|, \quad t>0, \tag{2.3}$$

and if $f\in B^r$, then

$$\omega_r^{\varphi}(f,t)\leqslant c(r)t^r\|\varphi^r f^{(r)}\|, \quad t>0. \tag{2.4}$$

We borrow from [13] the notion of the length of an interval $J:=[a,b]\subseteq I$, relative to its position in $I$. Namely,

$$/J/ := \frac{|J|}{\varphi((a+b)/2)}, \tag{2.5}$$

where $|J|:=b-a$. It follows from [13, (2.21)] that

$$\omega_k(f,|J|;J)\leqslant\omega_k^{\varphi}(f,/J/). \tag{2.6}$$

In our proof of the convex case we need the following lemma which, for the sake of convenience in its proof, we state in $[0,1]$.

**Lemma 2.1.** *Set* $\phi(x) := \sqrt{x(1-x)}$ *and* $\omega_k^\phi(f,t) := \omega_k^\phi(f,t;[0,1])$. *Then given* $k \geqslant 2$ *and* $f \in C[0,1]$, *the following holds for all* $0 < t \leqslant 1$:

$$\omega_k(f,t^2;[0,t^2]) \leqslant c(k)\omega_{k+1}^\phi(f,t) + c(k)t^{2k}|\Delta_{1/k}^k f(1/2)|.$$

**Proof.** We begin as in the proof of Marchaud inequality using divided differences. Recall that divided differences are defined by

$$[x_0;f] := f(x_0) \quad \text{and} \quad [x_0,\dots,x_k;f] := \frac{[x_0,\dots,x_{k-1};f]-[x_1,\dots,x_k;f]}{x_0-x_k}, \quad k \geqslant 1.$$

It is well known that for all $t_i \in [a,b]$, $i=0,\dots,k$, with $t_i \neq t_j$, $i \neq j$, and all $x_i \in [a,b]$, $i=0,\dots,k$, with $x_i \neq x_j$, $i \neq j$, we have

$$\begin{aligned}&|[t_0,\dots,t_k;f]-[x_0,\dots,x_k;f]|\\ &\quad \leqslant c\left(\min\left\{\min_{i\neq j}|t_i-t_j|, \min_{i\neq j}|x_i-x_j|\right\}\right)^{-k}\omega_{k+1}(f,b-a;[a,b]).\end{aligned} \tag{2.7}$$

Also, by Leviatan and Shevchuk [13, (2.25)]

$$\omega_k(f,t^2;[0,1]) \leqslant \omega_k^\phi(f,t), \quad k \geqslant 2. \tag{2.8}$$

We have to estimate $\Delta_h^k(f,x_0)$, where $0 < x_0 < t^2$ and $h > 0$ is such, that $x_0 \pm kh/2 \in [0,t^2]$, where without loss of generality we assume that $t^2 \leqslant 1/2k$. Let $l \in \mathbb{N}$, be defined by

$$2^l kh \leqslant \tfrac{1}{2} < 2^{l+1}kh. \tag{2.9}$$

Write $x^0 := x_0 - \frac{kh}{2}$, and for all $j=0,\dots,l$, denote

$$\delta_j := [x^0, x^0+2^j h, \dots, x^0+k2^j h; f].$$

Now, for all $j=0,\dots,l-1$, (2.7) yields

$$\begin{aligned}h^k|\delta_j-\delta_{j+1}| &\leqslant c2^{-jk}\omega_{k+1}(f,k2^{j+1}h;[x^0,x^0+k2^{j+1}h])\\ &\leqslant c2^{-jk}\omega_{k+1}(f,2^{j+2}t^2;[0,2^{j+2}t^2])\\ &\leqslant c2^{-jk}\omega_{k+1}^\phi(f,2^{1+j/2}t),\end{aligned}$$

where in the last inequality we applied (2.8). Therefore

$$\begin{aligned}h^k|\delta_j-\delta_{j+1}| &\leqslant c2^{-jk+(k+1)(1+j/2)}\omega_{k+1}^\phi(f,t)\\ &\leqslant c2^{-j/2}\omega_{k+1}^\phi(f,t),\end{aligned} \tag{2.10}$$

where we have used the fact that $k \geqslant 2$. Hence

$$\begin{aligned}|\Delta_h^k(f,x_0)| &= ch^k|\delta_0| \\ &\leqslant ch^k \sum_{j=0}^{l-1} |\delta_j - \delta_{j+1}| + ch^k|\delta_l| \\ &\leqslant c\omega_{k+1}^{\phi}(f,t) \sum_{j=0}^{\infty} 2^{-j/2} + ct^{2k}|\delta_l| \\ &= c\omega_{k+1}^{\phi}(f,t) + ct^{2k}|\delta_l|. \end{aligned} \tag{2.11}$$

Finally,

$$\begin{aligned} t^{2k}|\delta_l| &= |ct^{2k}\Delta_{1/k}^k f(1/2) + t^{2k}(\delta_l - [0,1/k,2/k,\dots,1;f])| \\ &\leqslant ct^{2k}|\Delta_{1/k}^k f(1/2)| + ct^{2k}\omega_{k+1}(f,1;[0,1]) \\ &\leqslant ct^{2k}|\Delta_{1/k}^k f(1/2)| + ct^{2k}\omega_{k+1}^{\phi}(f,1) \\ &\leqslant ct^{2k}|\Delta_{1/k}^k f(1/2)| + c\omega_{k+1}^{\phi}(f,t). \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12) we conclude that

$$|\Delta_h^k(f,x_0)| \leqslant t^{2k}|\Delta_{1/k}^k f(1/2)| + c\omega_{k+1}^{\phi}(f,t), \tag{2.13}$$

which completes the proof. □

Translating Lemma 2.1 to the interval $[-1,1]$, we immediately get

**Corollary 2.2.** *Given $k \geqslant 2$ and $f \in \mathbb{C}$, we have*

$$\omega_k(f,t^2;[-1,-1+t^2]) \leqslant c(k)\omega_{k+1}^{\varphi}(f,t) + c(k)t^{2k}|\Delta_{2/k}^k f(0)|, \tag{2.14}$$

*and by symmetry*

$$\omega_k(f,t^2;[1-t^2,1]) \leqslant c(k)\omega_{k+1}^{\varphi}(f,t) + c(k)t^{2k}|\Delta_{2/k}^k f(0)|. \tag{2.15}$$

Next, we construct convex polynomials on any given interval such that they are close to a convex function there, and we construct polynomials which change convexity once on a given interval and again stay close to a function which changes convexity once there. Eventually, these two types of polynomials will provide the pieces we glue together in order to obtain the piecewise polynomials required by Theorem LS.

**Lemma 2.3.** *Let $k \geqslant 1$ and let $f \in \mathbb{C}^2[0,1]$, be convex and such that $f(0) = f'(0) = 0$, and*

$$\omega_k(f'',1) = 1. \tag{2.16}$$

*Then there exists a convex polynomial $P \in \Pi_{k+1}$, satisfying $P(0) = f(0)$, and $P(1) = f(1)$, and either $P'(0) = f'(0)$ and $P'(1) \leqslant f'(1)$, or $P'(0) \geqslant f'(0)$ and $P'(1) = f'(1)$, such* *that*

$$\|f - P\|_{[0,1]} \leqslant c, \tag{2.17}$$

*where* $c = c(k)$.

Note that if $\omega_k(f'', 1) = 0$, then we may take $P = f$. Otherwise (2.16) is just a normalization.

**Proof.** By virtue of [12, Lemma 2] there exists a nondecreasing polynomial $p \in \Pi_k$, such that $p(0) = f'(0)$ and $p(1) = f'(1)$ and

$$\|f' - p\|_{[0,1]} \leqslant c(k). \tag{2.18}$$

Let

$$P_*(x) := \int_0^x p(u)\, du \in \Pi_{k+1}.$$

Then $P_*$ is convex, and since $p(0) = 0$ and $p$ is nondecreasing, it is nonnegative and nondecreasing. Also, by (2.18),

$$\|f - P_*\|_{[0,1]} \leqslant \int_0^1 |f'(u) - p(u)|\, du \leqslant c. \tag{2.19}$$

Now, if $P_*(1) \geqslant f(1)$, (note that by virtue of (2.16), $f(1) > 0$), then set

$$P := \frac{f(1)}{P_*(1)} P_*.$$

Then $P$ is convex, $P(0) = 0 = f(0)$, $P'(0) = \frac{f(1)}{P_*(1)} p(0) = 0 = f'(0)$, and $P(1) = f(1)$. Finally,

$$P'(1) = \frac{f(1)}{P_*(1)} P'_*(1) \leqslant P'_*(1) = p(1) = f'(1),$$

and by (2.19),

$$|f(x) - P(x)| \leqslant |f(x) - P_*(x)| + \left| \frac{P_*(x)}{P_*(1)} (P_*(1) - f(1)) \right| \leqslant 2c,$$

where we used the fact that $P_*(x) \leqslant P_*(1)$ since $P_*$ is nondecreasing. Hence (2.17) is proved.

Otherwise, $P_*(1) < f(1)$. Observe that

$$f'(1) - f(1) = \int_0^1 u f''(u)\, du \geqslant 0,$$

so that we may set

$$P(x) := P_*(x) + \frac{f(1) - P_*(1)}{f'(1) - P_*(1)} (x f'(1) - P_*(x)).$$

Then $P(0) = 0 = f(0)$, $P(1) = f(1)$ and $P'(1) = f'(1)$, where for the last equality we applied $P'_*(1) = p(1) = f'(1)$. Also

$$P'(0) = \frac{f(1) - P_*(1)}{f'(1) - P_*(1)} f'(1) \geqslant 0$$

and

$$P''(x) = \frac{f'(1) - f(1)}{f'(1) - P_*(1)} p'(x) \geqslant 0, \quad 0 \leqslant x \leqslant 1.$$

Finally, by (2.19),

$$|f(x) - P(x)| \leqslant |f(x) - P_*(x)| + (f(1) - P_*(1)) \frac{xf'(1) - P_*(x)}{f'(1) - P_*(1)}$$
$$\leqslant 2c,$$

where we used the fact that $xf'(1) - P_*(x)$ is nondecreasing in $[0,1]$, hence $0 \leqslant xf'(1) - P_*(x) \leqslant f'(1) - P_*(1)$. Indeed, by virtue of the monotonicity of $p(x)$,

$$(xf'(1) - P_*(x))' = f'(1) - p(x) = p(1) - p(x) \geqslant 0.$$

Again (2.17) is proved. □

An immediate consequence is

**Corollary 2.4.** *Let $k \geqslant 1$ and let $f \in \mathbb{C}^2[a, a+h]$, $h > 0$, be convex. Then there exists a convex polynomial $P \in \Pi_{k+1}$ satisfying $P(a) = f(a)$ and $P(a+h) = f(a+h)$, and either $P'(a) = f'(a)$ and $P'(a+h) \leqslant f'(a+h)$, or $P'(a) \geqslant f'(a)$ and $P'(a+h) = f'(a+h)$, such that*

$$\|f - P\|_{[a,a+h]} \leqslant ch^2 \omega_k(f'', h; [a, a+h]), \tag{2.20}$$

*where $c = c(k)$.*

**Lemma 2.5.** *Let $k \geqslant 1$ and let $0 < \beta < 1$ be fixed. Assume that $f \in \mathbb{C}^2[0,1]$ is such that*

$$f''(x)(x - \beta) \geqslant 0, \quad 0 \leqslant x \leqslant 1.$$

*If $P_{k-1} \in \Pi_{k-1}$ satisfies*

$$P_{k-1}(x)(x - \beta) \geqslant 0, \quad 0 \leqslant x \leqslant 1, \tag{2.21}$$

*then there exists an $\alpha$ such that the polynomial*

$$P_{k+1}(x) := \alpha x + f(0) + \int_0^x (x - u) P_{k-1}(u)\, du,$$

*satisfies either*

$$P'_{k+1}(0) = f'(0) \quad \text{and} \quad P'_{k+1}(1) \leqslant f'(1), \tag{2.22}$$

*or*

$$P'_{k+1}(0) \leqslant f'(0) \quad \text{and} \quad P'_{k+1}(1) = f'(1), \tag{2.23}$$

*and*

$$\|f - P_{k+1}\|_{[0,1]} \leqslant \tfrac{3}{2}\|f'' - P_{k-1}\|_{[0,1]}. \tag{2.24}$$

*Note that by* (2.21)

$$P''_{k+1}(x)(x-\beta) = P_{k-1}(x)(x-\beta) \geqslant 0. \tag{2.25}$$

**Proof.** Set $P_k(x) := \int_0^x P_{k-1}(u)\,du$, and let

$$\alpha := \begin{cases} f'(0) & \text{if } P_k(1) + f'(0) \leqslant f'(1), \\ f'(1) - P_k(1) & \text{otherwise.} \end{cases}$$

Since $P'_{k+1}(0) = \alpha$ and $P'_{k+1}(1) = \alpha + P_k(1)$, then either (2.22) or (2.23) is self-evident. In order to prove (2.24) we observe that

$$f(x) = xf'(0) + f(0) + \int_0^x (x-u)f''(u)\,du,$$

whence

$$|f(x) - P_{k+1}(x)| \leqslant |f'(0) - \alpha| + \tfrac{1}{2}\|f'' - P_{k-1}\|_{[0,1]} \leqslant \tfrac{3}{2}\|f'' - P_{k-1}\|_{[0,1]},$$

where for the right-hand inequality we applied that either $f'(0) - \alpha = 0$ or $f'(0) - \alpha = \int_0^1 (P_{k-1}(u) - f''(u))\,du$. □

Again the following consequence is readily seen

**Corollary 2.6.** *Let $k \geqslant 1$ and let $a < \beta < a+h$ be fixed and assume that $f \in \mathbb{C}^2[a, a+h]$ is such that*

$$f''(x)(x-\beta) \geqslant 0, \quad a \leqslant x \leqslant a+h.$$

*If $P_{k-1} \in \Pi_{k-1}$ satisfies*

$$P_{k-1}(x)(x-\beta) \geqslant 0, \quad a \leqslant x \leqslant a+h,$$

*then there exists an $\alpha$ such that the polynomial*

$$P_{k+1}(x) := \alpha(x-a) + f(a) + \int_a^x (x-u)P_{k-1}(u)\,du,$$

*satisfies either*

$$P'_{k+1}(a) = f'(a) \quad \text{and} \quad P'_{k+1}(a+h) \leqslant f'(a+h),$$

*or*

$$P'_{k+1}(a) \leqslant f'(a) \quad \text{and} \quad P'_{k+1}(a+h) = f'(a+h),$$

*and*

$$\|f - P_{k+1}\|_{[a,a+h]} \leqslant \tfrac{3}{2}h^2\|f'' - P_{k-1}\|_{[a,a+h]}.$$

## 3. Convex approximation

In 1994, Hu et al. [5] and Kopotun [7] independently proved that there exists an absolute constant $C$, such that for every $f \in \Delta^2$,

$$E_n^{(2)}(f) \leqslant C\omega_3(f, 1/n), \quad n \geqslant 2. \tag{3.1}$$

By virtue of (2.1), inequality (3.1) readily implies for $f \in \mathbb{C}^r$,

$$E_n^{(2)}(f) \leqslant \frac{C}{n^r} \omega_k(f^{(r)}, 1/n), \quad n \geqslant 2, \tag{3.2}$$

for all $k + r \leqslant 3$, and thus contains results for $r = 0$ of [1] (for $k = 1$), and of [19] (for $k = 2$).

For the degree of unconstrained polynomial approximation,

$$E_n(f) := \inf_{p_n \in \Pi_n} \|f - p_n\|,$$

we have the well known Jackson estimates, namely, if $f \in \mathbb{C}$, then

$$E_n(f) \leqslant c(k)\omega_k^\varphi(f, 1/n), \quad n \geqslant k - 1, \quad k = 1, 2, \dots, \tag{3.3}$$

hold, and imply that if $f \in \mathbb{C}^{(r)}$, then

$$E_n(f) \leqslant \frac{c(k, r)}{n^r} \omega_k^\varphi(f^{(r)}, 1/n), \quad n \geqslant k + r - 1. \tag{3.4}$$

In particular if $f \in W^r$, then

$$E_n(f) \leqslant \frac{c(r)}{n^r} \|f^{(r)}\|, \quad n \geqslant r - 1, \tag{3.5}$$

and if $f \in B^r$, then

$$E_n(f) \leqslant \frac{c(r)}{n^r} \|\varphi^r f^{(r)}\|, \quad n \geqslant r - 1. \tag{3.6}$$

However, the situation in constrained approximation is much more involved. For instance, Wu and Zhou [20] established the existence of an $f \in \Delta^2 \cap \mathbb{C}^1$ such that

$$\limsup_{n \to \infty} \frac{nE_n^{(2)}(f)}{\omega_4(f', 1/n)} = \infty.$$

Hence, for $k \geqslant 5$, the estimate

$$E_n^{(2)}(f) \leqslant A\omega_k(f, 1/n), \quad n \geqslant N, \tag{3.7}$$

is not valid for all $f \in \Delta^2$, even if we allow the constants $A$ and $N$ to depend on $f$ (cf. (3.3)). Wu and Zhou [20] have conjectured that (3.7) cannot be gotten (with constants $A$ and $N$ that depend on $f$) even for $k = 4$. This is in view of an earlier proof of Shvedov [19] that for each $n \geqslant 1$ and any $A > 0$, there exists an $f := f_{A,n} \in \Delta^2$ for which

$$E_n^{(2)}(f) > A\omega_4(f, 1/n).$$

We first disprove this conjecture, that is, we show that (3.7) is valid for $k=4$ with an absolute constant $C$ provided we allow $N=N(f)$. Specifically, we prove a little more, namely,

**Theorem 3.1.** *If $f\in\Delta^2$, then*

$$E_n^{(2)}(f)\leqslant C\omega_4^{\varphi}(f,1/n)+\frac{C}{n^6}|\Delta_{2/3}^3 f(0)| \\ \leqslant C\omega_4^{\varphi}(f,1/n)+\frac{C}{n^6}\|f\|, \quad n>1. \tag{3.8}$$

An immediate consequence is

**Corollary 3.2.** *There exists an absolute constant $C$, such that for every $f\in\Delta^2$ there is an $N=N(f)$ for which*

$$E_n^{(2)}(f)\leqslant C\omega_4^{\varphi}(f,1/n)\leqslant C\omega_4(f,1/n), \quad n\geqslant N. \tag{3.9}$$

Note that if $\omega_4^{\varphi}(f,1/n)=0$ for some $n$, then $f$ is a polynomial of degree $\leqslant 3$. Thus $E_n^{(2)}(f)=0$, $n\geqslant 3$, and $E_2^{(2)}(f)=E_2(f)=9\times 2^{-6}|\Delta_{2/3}^3 f(0)|$. Therefore Theorem 3.1 and Corollary 3.2 remain valid in this case.

**Remark.** It is interesting to point out that if $f$ is even, then $\Delta_{2/3}^3 f(0)=0$. Hence for even functions (3.9) actually holds for all $n>1$.

Recalling that previous positive estimates by Mania and Shevchuk (see [17]) for $r\geqslant 2$ yield for every $k\geqslant 1$,

$$E_n^{(2)}(f)\leqslant \frac{c}{n^r}\omega_k(f^{(r)},1/n), \quad n\geqslant N, \tag{3.10}$$

where $c=c(k,r)$ and $N=N(k,r)=k+r-1$, while by Mania (see [17]), (3.10) cannot be gotten for $r=1$ and $k\geqslant 3$, (cf. (3.4)), we may now summarize the results in the following array: where the symbol $+$ stands for cases $(k,r)$ for which (3.10) holds with constants $c$ and $N$ which may depend only on $k$ and $r$, the symbol $\ominus$ indicates that (3.10) is invalid with constants as above, but is valid if we allow either $c$ or $N$ to depend on $f$ itself, and finally the symbol $-$, states that (3.10) cannot in general be gotten (Fig. 1). The case $k=0$ describes the validity of the estimate

$$E_n^{(2)}(f)\leqslant \frac{c(r)}{n^r}\|f^{(r)}\|, \quad n\geqslant r-1 \tag{3.11}$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $r$ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋰ |
| 2 | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ | ⋯ |
| 1 | $+$ | $+$ | $+$ | $\ominus$ | $-$ | $-$ | ⋯ |
| 0 | | $+$ | $+$ | $+$ | $\ominus$ | $-$ | ⋯ |
| | 0 | 1 | 2 | 3 | 4 | 5 | $k$ |

Fig. 1.

for every $f \in W^r \cap \Delta^2$, $r \geqslant 1$, which readily follows from (2.3) and the validity of (3.10) for $k = 1$ and $r \geqslant 0$ (cf. (3.5)).

We would like to point out another consequence of Theorem 3.1, before proceeding to prove it. It follows from (2.4) that

**Corollary 3.3.** *Let $f \in B^4 \cap \Delta^2$. Then*

$$E_n^{(2)}(f) \leqslant \frac{C}{n^4}\|\varphi^4 f^{(4)}\| + \frac{C}{n^6}\|f\|, \quad n \geqslant 1.$$

*Consequently, there exists an $N = N(f)$ for which*

$$E_n^{(2)}(f) \leqslant \frac{C}{n^4}\|\varphi^4 f^{(4)}\|, \quad n \geqslant N. \tag{3.12}$$

It has long been known that the inequality

$$E_n^{(2)}(f) \leqslant \frac{c(r)}{n^r}\|\varphi^r f^{(r)}\|, \quad n \geqslant r-1, \tag{3.13}$$

is valid for $r \neq 4$ (cf. (3.6) and (3.11)). It is due to Leviatan [10] for $r = 1, 2$, and to Kopotun [6] for $r = 3$ and $r \geqslant 5$. (In fact for $r \leqslant 3$, the more general estimate

$$E_n^{(2)}(f) \leqslant C\omega_k^{\varphi}(f, 1/n), \quad 1 \leqslant k \leqslant 3,$$

was first proved by Leviatan [10] for $k = 2$, and later by Kopotun [7] for $k = 3$, (see also [8]).

Moreover, for $r = 4$, in general (3.13) cannot be gotten for any fixed $n$, since Kopotun [6] has proved that for each $n \geqslant 1$ and any $A > 0$, there exists a function $f = f_{n,A} \in B^4 \cap \Delta^2$ such that

$$E_n^{(2)}(f) > \frac{A}{n^4}\|\varphi^r f^{(4)}\|.$$

However, note again that for even functions (3.12) holds for $n > 1$ (see Remark after Corollary 3.2).

**Proof of Theorem 3.1.** Recall the Chebyshev partition $-1 = x_n < x_{n-1} < \cdots < x_1 < x_0 = 1$, and $I_i := [x_i, x_{i-1}]$, $1 \leqslant i \leqslant n$. Denote $J_1 = J_2 := I_3 \cup I_2 \cup I_1$, $J_n = J_{n-1} := I_{n-2} \cup I_{n-1} \cup I_n$, and $J_i := \bigcup_{j=i-2}^{i+2} I_j$, $3 \leqslant i \leqslant n-2$. For a given $f \in \Delta^2$, Shevchuk [18], constructed a continuous piecewise cubic polynomial $S \in \Delta^2$, on the Chebyshev partition, such that $S$ interpolates $f$ on the partition, and

$$\|f - S\|_{I_i} \leqslant C\omega_4(f, |J_i|; J_i), \quad 3 \leqslant i \leqslant n-2,$$

$$\|f - S\|_{I_i} \leqslant C\omega_3(f, |J_i|; J_i), \quad i = 1, 2, n-1, n. \tag{3.14}$$

For the Chebyshev partition we obtain from (2.5) that $/J_i/ \leqslant \frac{C}{n}$. Hence by virtue of (2.6), (3.14) implies

$$\|f - S\|_{I_i} \leqslant C\omega_4^{\varphi}(f, 1/n), \quad 3 \leqslant i \leqslant n-2. \tag{3.15}$$

At the same time we observe that $J_1 = [1 - A/n^2, 1]$, with $A = A(n) \leqslant C$, and similarly for $J_2$. Also $J_n = [-1, -1 + A/n^2]$, with $A = A(n) \leqslant C$, and similarly for

$J_{n-1}$. Thus by (2.14) and (2.15) we conclude that (3.14) yields

$$\|f - S\|_{I_i} \leqslant C\omega_4^{\varphi}(f, 1/n) + \frac{C}{n^6}|\Delta_{2/3}^3 f(0)|, \quad i = 1, 2, n-1, n. \tag{3.16}$$

Combining (3.15) and (3.16) we obtain

$$\omega_4^{\varphi}(S, 1/n) \leqslant C\omega_4^{\varphi}(f, 1/n) + \frac{C}{n^6}|\Delta_{2/3}^3 f(0)|, \tag{3.17}$$

which together with Theorem LS completes the proof of Theorem 3.1. □

## 4. Coconvex approximation

In this section, we are dealing with functions that change convexity at least once in $[-1, 1]$, i.e., $s \geqslant 1$. Given $Y_s \in \mathbb{Y}_s$, we wish to investigate the validity of the estimates

$$E_n^{(2)}(f, Y_s) \leqslant \frac{c}{n^r}\omega_k^{\varphi}(f^{(r)}, 1/n), \quad n \geqslant N, \tag{4.1}$$

for functions $f \in \Delta^2(Y_s) \cap \mathbb{C}^r$, $r \geqslant 0$, and that of

$$E_n^{(2)}(f, Y_s) \leqslant \frac{c}{n^r}\|f^{(r)}\|, \quad n \geqslant N, \tag{4.2}$$

for functions $f \in \Delta^2(Y_s) \cap W^r$, $r \geqslant 1$.

Recently, Kopotun et al. [9] have proved the validity of (4.1) for all pairs $(k, r)$, $k + r \leqslant 3$, with a constant $c = c(s)$, and with $N = N(Y_s)$. Moreover, if $s = 1$ and $k + r \leqslant 2$, then (4.1) holds for all $n \geqslant 1$ (see [15]). However, if $r = 1$ and $k = 2$, and consequently also if $r = 0$ and $k = 3$, then Pleshakov and Shatalina [16] proved that $N(Y_s)$ may not be replaced by $N(s)$.

In fact there are known quite a few negative results. The first, which even preceded [16], is due to Wu and Zhou [20] who proved that for $s \geqslant 1$, for each $k > 2$ and any $Y_s \in \mathbb{Y}_s$, there exists an $f \in \Delta^2(Y_s) \cap \mathbb{C}^1$, such that

$$\limsup_{n\to\infty} \frac{nE_n^{(2)}(f, Y_s)}{\omega_k(f', 1/n)} = \infty. \tag{4.3}$$

Therefore, (4.1) cannot be had for $r = 1$ and any $k > 2$, even with constants $c$ and $N$ which depend on $f$. Moreover, by virtue of (2.1), (4.1) cannot be gotten for $r = 0$ and any $k > 3$, again even with constants $c$ and $N$ which depend on $f$. Very recently Gilewicz and Yushchenko [4], have extended (4.3), proving that for each $k > 3$ and any $Y_s \in \mathbb{Y}_s$, there exists an $f \in \Delta^2(Y_s) \cap \mathbb{C}^2$, such that

$$\limsup_{n\to\infty} \frac{n^2 E_n^{(2)}(f, Y_s)}{\omega_k(f'', 1/n)} = \infty. \tag{4.4}$$

Note that by virtue of (2.1), (4.4) implies (4.3) but only for $k > 4$. Again, this shows that (4.1) cannot be gotten for $r = 2$ and any $k > 3$, even with constants $c$ and $N$ which depend on $f$.

Also, Leviatan and Shevchuk [15], extending the result of Pleshakov and Shatalina [16], showed that if $s \geqslant 2$, then (4.1) cannot be gotten with $c = c(k,r,s)$ and $N = N(k,r,s)$, for any $r = 0,1,2,3$ with any $k \geqslant 1$.

Our aim here is to prove that the answer is affirmative in all remaining cases, that is, we prove two theorems.

**Theorem 4.1.** *If $f \in \Delta^2(Y_s) \cap \mathbb{C}^2$, then for each $k \leqslant 3$,*

$$E_n^{(2)}(f, Y_s) \leqslant \frac{c}{n^2}\omega_k^{\varphi}(f'', 1/n) \leqslant \frac{c}{n^2}\omega_k(f'', 1/n), \quad n \geqslant N, \tag{4.5}$$

*where $c = c(s)$ and $N = N(Y_s)$. Furthermore, if $s = 1$ and $k \leqslant 2$, then $N = k + 1$.*

**Theorem 4.2.** *Let $r \geqslant 3$ and assume that $f \in \Delta^2(Y_s) \cap \mathbb{C}^r$. Then* (4.1) *holds for each $k \geqslant 1$, with constants $c = c(k,r,s)$ and $N = N(k,r,Y_s)$. Furthermore, if $s = 1$, then* (4.1) *holds with $N = k + r - 1$.*

An immediate consequence of the affirmative results is an affirmative answer to the question of the validity of (4.2), namely,

**Corollary 4.3.** *If $f \in \Delta^2(Y_s) \cap W^r$, $r \geqslant 1$, then* (4.2) *holds for $c = c(r,s)$, and $N = N(r, Y_s)$ if $s \geqslant 2$, and $N = r - 1$ if $s = 1$.*

Also, standard technique enables one to exchange the roles of $c$ and $N$ in the above theorems. Namely, we can state

**Corollary 4.4.** *If $f \in \Delta^2(Y_s) \cap \mathbb{C}^2$, then for each $k \leqslant 3$,*

$$E_n^{(2)}(f, Y_s) \leqslant \frac{A}{n^2}\,\omega_k(f'', 1/n), \quad n \geqslant k + 1,$$

*where $A = A(Y_s)$.*

**Corollary 4.5.** *Let $r \geqslant 3$ and assume that $f \in \Delta^2(Y_s) \cap \mathbb{C}^r$. Then* (4.1) *holds with a constant $A = A(k,r,Y_s)$, for each $k \geqslant 1$, and all $n \geqslant k + r - 1$.*

We are in a position to summarize the positive and negative results in two separate truth tables, one for $s = 1$ (Fig. 2), and the other for $s \geqslant 2$ (Fig. 3), where the symbol $+$ stands for cases $(k,r)$ for which (4.1) and (4.2) hold with a constant $c$ which may depend on $k$ and $r$, and $N = k + r - 1$, the symbol $\oplus$ indicates that (4.1) is invalid with constants as above, but is valid if we allow either $c$ or $N$ to depend on $Y_s$, and finally the symbol $-$, states that (4.1) cannot in general be gotten.

Note that by Theorem 4.2 we know that (4.1) holds at least with $N = N(k,r,Y_s)$, and that when $s \geqslant 2$, this cannot be improved for any $r \leqslant 3$. In a forthcoming paper with K. Kopotun, it will be proved that for $s \geqslant 2$, one cannot replace any of the $\oplus$'s by the symbol $+$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $r$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\iddots$ |
| 3 | $+$ | $+$ | $+$ | $+$ | $+$ | $\cdots$ |
| 2 | $+$ | $+$ | $+$ | $\oplus$ | $-$ | $\cdots$ |
| 1 | $+$ | $+$ | $\oplus$ | $-$ | $-$ | $\cdots$ |
| 0 | | $+$ | $+$ | $\oplus$ | $-$ | $\cdots$ |
| | 0 | 1 | 2 | 3 | 4 | $k$ |

Fig. 2. $s = 1$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $r$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\iddots$ |
| 3 | $\oplus$ | $\oplus$ | $\oplus$ | $\oplus$ | $\oplus$ | $\cdots$ |
| 2 | $\oplus$ | $\oplus$ | $\oplus$ | $\oplus$ | $-$ | $\cdots$ |
| 1 | $\oplus$ | $\oplus$ | $\oplus$ | $-$ | $-$ | $\cdots$ |
| 0 | | $\oplus$ | $\oplus$ | $\oplus$ | $-$ | $\cdots$ |
| | 0 | 1 | 2 | 3 | 4 | $k$ |

Fig. 3. $s \geqslant 2$.

**Proof of Theorems 4.1 and 4.2.** Given $f \in \Delta^2(Y_s) \cap \mathbb{C}^r$, $r \geqslant 2$, we take $N(Y_s)$ so big that if $n \geqslant N$, then for each $1 \leqslant i \leqslant s$, the set $O_i$ defined in Section 1, contains only one $y_i$, and $O_i$ and $O_{i+1}$, $1 \leqslant i \leqslant s-1$, are separated by at least one interval of the partition. Thus, we have no restriction on $N$, if $s = 1$. Then we have $s$ intervals $O_i =: (a_i, b_i)$, $i = 1, \dots, s$ such that either

$$f''(x)(x - y_i) \geqslant 0, \quad a_i < x < b_i \tag{4.6}$$

or

$$f''(x)(x - y_i) \leqslant 0, \quad a_i < x < b_i. \tag{4.7}$$

We first deal with the case $f \in \Delta^2(Y_s) \cap \mathbb{C}^2$. As per Theorem 4.1, we only have to consider $k \leqslant 3$, and we define polynomials $P_{k-1,i} \in \Pi_{k-1}$, $k = 1, 2, 3$, which satisfy, respectively,

$$P_{k-1,i}(x)(x - y_i) \geqslant 0, \quad a_i < x < b_i \tag{4.8}$$

or

$$P_{k-1,i}(x)(x - y_i) \leqslant 0, \quad a_i < x < b_i, \tag{4.9}$$

and are close to $f''$. To this end we take $P_{0,i} \equiv 0$, $P_{1,i}$ to be the linear polynomial interpolating $f''$ at $y_i$ and at $a_i$ or $b_i$ whichever is farther from $y_i$, and finally $P_{2,i}$ to be the quadratic polynomial interpolating $f''$ at $a_i$, $y_i$ and $b_i$. By Whitney's theorem we know that

$$\|f'' - P_{k-1,i}\|_{O_i} \leqslant C\omega_k(f'', |O_i|; O_i), \quad k = 1, 2, 3, \tag{4.10}$$

where $C$ depends on the ratios between $|O_i|$ and the distances between the points of interpolation. Thus $C$ is an absolute constant for $k = 1, 2$, but for $k = 3$ one has to worry about either $y_1$ or $y_s$ being too close to one of the endpoints (this would make $y_1$ too close to $b_1$ and $y_s$ too close to $a_s$). In order to overcome this problem and have an absolute constant $C$ also when $k=3$, we have to take $n \geqslant N = N(Y_s)$ even when $s=1$.

When $f \in \Delta^2(Y_s) \cap \mathbb{C}^r$, $r \geqslant 3$, we apply [3, Corollary 3.1] to $f''$ and $r=1$, and obtain for each $k \geqslant 2$, the existence of $P_{k-1,i} \in \Pi_{k-1}$ such that (4.8) and (4.9) hold, respectively, and

$$\|f'' - P_{k-1,i}\|_{O_i} \leqslant c|O_i|\omega_{k-1}(f^{(3)}, |O_i|; O_i). \tag{4.10$'$}$$

Thus, in all cases we conclude by Corollary 2.6 and (4.8) and (4.9), that there exists a polynomial $P_{k+1,i} \in \Pi_{k+1}$ which is coconvex with $f$ on $O_i$, $P_{k+1,i}(a_i) = f(a_i) + \alpha_i$, where $\alpha_i$ is an arbitrary constant to be prescribed, and such that

$$\|f - P_{k+1,i}\|_{O_i} \leqslant |\alpha_i| + \tfrac{3}{2}|O_i|^2 \|f'' - P_{k-1,i}\|_{O_i}, \tag{4.11}$$

where by (4.10) and (4.10$'$) we have an estimate on the second term on the right. Note that (4.11) implies that

$$|P_{k+1,i}(b_i) - f(b_i)| \leqslant |\alpha_i| + \tfrac{3}{2}|O_i|^2 \|f'' - P_{k-1,i}\|_{O_i}. \tag{4.12}$$

Also if (4.6) holds, then

$$P'_{k+1,i}(a_i) \leqslant f'(a_i) \quad \text{and} \quad P'_{k+1,i}(b_i) \leqslant f'(b_i), \tag{4.13}$$

and if (4.7) holds, then

$$P'_{k+1,i}(a_i) \geqslant f'(a_i) \quad \text{and} \quad P'_{k+1,i}(b_i) \geqslant f'(b_i). \tag{4.14}$$

In all other intervals $I_j$, $j \in H$ (see Section 1), $f$ is either convex in $I_j$ or $f$ is concave there. If $g_j := f + \beta_j$, where $\beta_j$ is an arbitrary constant to be prescribed, then by Corollary 2.4, there exists a polynomial $p_{k+1,j} \in \Pi_{k+1}$, coconvex with $f$ and satisfying $p_{k+1,j}(x_j) = g_j(x_j)$ and $p_{k+1,j}(x_{j-1}) = g_j(x_{j-1})$. Also if $f$ is convex, then we have

$$p'_{k+1,j}(x_j) \geqslant f'(x_j) \quad \text{and} \quad p'_{k+1,j}(x_{j-1}) \leqslant f'(x_{j-1}), \tag{4.15}$$

and if $f$ is concave, then

$$p'_{k+1,j}(x_j) \leqslant f'(x_j) \quad \text{and} \quad p'_{k+1,j}(x_{j-1}) \geqslant f'(x_{j-1}). \tag{4.16}$$

Finally by (2.2)

$$\begin{aligned}\|f - p_{k+1,j}\|_{I_j} &\leqslant |\beta_j| + c|I_j|^2 \omega_k(f'', |I_j|; I_j)\\ &\leqslant |\beta_j| + cn^{-2}\omega_k^{\varphi}(f'', 1/n),\end{aligned} \tag{4.17}$$

since $/I_j/ \leqslant C/n$.

We now construct the piecewise polynomial $S \in \Sigma_{k+2,n}(Y_s) \cap \Delta^2(Y_s)$, sweeping $[-1,1]$ from left to right. Let $a_s = x_{j_0}$, where $O_s = (a_s, b_s)$, and let $\alpha_s := 0$. Then for $j_0 < j \leqslant n$, we take $\beta_j = 0$ and set

$$S_{|_{I_j}} := p_{k+1,j}, \quad j_0 < j \leqslant n,$$

and

$$S_{|_{O_s}} := P_{k+1,s}.$$

Note that $S$ is continuous in $[-1, b_s)$, and by (4.14) and (4.15), or (4.13) and (4.16), respectively, it is coconvex with $f$ there. Suppose that we have defined $S$ in $[-1, b_i)$, $1 < i \leqslant s$, let $b_i = x_{j_1}$ and $a_{i-1} = x_{j_2}$. Then we take $\alpha_{i-1} := \sum_{m=i}^{s}(P_{k+1,m}(b_m) - f(b_m))$, and for $j_2 < j \leqslant j_1$, $\beta_j := \alpha_{i-1}$. Then we set

$$S_{|_{I_j}} := p_{k+1,j}, \quad j_2 < j \leqslant j_1$$

and

$$S_{|_{O_{i-1}}} := P_{k+1,i-1}.$$

This guarantees that $S$ is continuous in $[-1, b_{i-1})$ and coconvex with $f$ there. Finally if $b_1 = x_{j_3}$, then for $1 \leqslant j \leqslant j_3$, we take $\beta_j := \sum_{m=1}^{s}(P_{k+1,m}(b_m) - f(b_m))$, and we set

$$S_{|_{I_j}} := p_{k+1,j}, \quad 1 \leqslant j \leqslant j_3.$$

It is readily seen that we have obtained an $S \in \Sigma_{k+2,n}(Y_s) \cap \Delta^2(Y_s)$.

Again, we deal first with $f \in \mathbb{C}^2$. Since $/O_i/ \leqslant C/n$, it follows by (4.10) that

$$\|f'' - P_{k-1,i}\|_{O_i} \leqslant C\omega_k^{\varphi}(f'', 1/n), \quad k = 1, 2, 3.$$

Hence, combining with (4.11), (4.12) and (4.17), yields

$$\|f - S\| \leqslant Csn^{-2}\omega_k^{\varphi}(f'', 1/n), \quad k = 1, 2, 3. \tag{4.18}$$

This in turn implies

$$\begin{aligned}\omega_{k+2}^{\varphi}(S, 1/n) &\leqslant \omega_{k+2}^{\varphi}(f, 1/n) + Csn^{-2}\omega_k^{\varphi}(f'', 1/n) \\ &\leqslant cn^{-2}\omega_k^{\varphi}(f'', 1/n), \quad k = 1, 2, 3. \end{aligned} \tag{4.19}$$

Therefore, we apply (4.18), (4.19), and Theorem LS to obtain a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\|f - P_n\| \leqslant cn^{-2}\omega_k^{\varphi}(f'', 1/n), \quad k = 1, 2, 3, \quad n \geqslant N. \tag{4.20}$$

This completes the proof of (4.5) with $c = c(s)$ and $N = N(Y_s)$. If $s = 1$ and $k = 1, 2$, then so far we have imposed no restriction on $N$, except for what is implied by Theorem LS, namely, that $N \geqslant c_*(k)$. By the constrained Whitney inequalities due to Pleshakov and Shatalina [16], we may take $N = k + 1$. Thus Theorem 4.1 is proven.

Now we assume that $f \in \mathbb{C}^r$, $r \geqslant 3$ and let $k \geqslant 2$. Then it follows by (4.10′) that

$$\|f'' - P_{k-1,i}\|_{O_i} \leqslant cn^{-1}\omega_{k-1}^{\varphi}(f^{(3)}, 1/n).$$

Hence, combining with (4.11), (4.12) and (4.17), yields

$$\|f - S\| \leqslant csn^{-3}\omega_{k-1}^{\varphi}(f^{(3)}, 1/n). \tag{4.18′}$$

This in turn gives

$$\begin{aligned}\omega_{k+2}^{\varphi}(S, 1/n) &\leqslant \omega_{k+2}^{\varphi}(f, 1/n) + csn^{-3}\omega_{k-1}^{\varphi}(f^{(3)}, 1/n) \\ &\leqslant cn^{-3}\omega_{k-1}^{\varphi}(f^{(3)}, 1/n), \end{aligned} \tag{4.19′}$$

where $c = c(k, s)$. Therefore, we apply (4.18′), (4.19′), and Theorem LS to obtain a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\|f - P_n\| \leqslant cn^{-3}\omega_{k-1}^{\varphi}(f^{(3)}, 1/n), \quad n \geqslant N. \tag{4.20′}$$

Since $f \in \mathbb{C}^r$, $r \geqslant 3$, it follows by (2.2) that (4.1) is valid for all $r \geqslant 3$ and $k \geqslant 1$, with $c = c(k, r, s)$ and $N = N(k, r, Y_s)$. For $s = 1$, we so far have imposed no restriction on $N$, except for what is implied by Theorem LS, namely, that $N \geqslant c_*(k, r)$. Again, by the constrained Whitney inequalities of Pleshakov and Shatalina [16], we may take $N = k + r - 1$. Theorem 4.2 is proved. □

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